Regarding the absolute stability of Störmer-Cowell methods

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Abstract

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Regarding the absolute stability of Störmer-Cowell methods

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Abstract

Störmer-Cowell methods, a popular class of methods for computations in celestial mechanics, is known to exhibit orbital instabilities when the order of the methods exceed two. Analysing the absolute stability of Störmer-Cowell methods close to zero we present a characterization of these instabilities for methods of all orders.

1 Introduction

Many mechanical problems in physics are expressed as second order differential equations of the form,

$$y'' = f(x, y), \qquad y(x_0) = y_0, \quad y'(x_0) = y_{1,0}. \tag{1}$$

Numerical algorithms for such equations are arguably as old as mechanics itself: Isaac Newton was the inventor of what we today call the Störmer-Verlet scheme[5], one of the most popular schemes for mechanical problems. Carl Störmer, eponymous to the Störmer-Verlet method, published his methods aimed at computing orbits of charged particles in Earth's magnetic field(aurora borealis) as early as in 1907. Störmer's methods is a class of multistep methods containing the Verlet method as a simplest case. A related class of methods is Cowell's methods, dating back to 1910. These two classes of methods, Störmer's and Cowell's or predictor-corrector pair of these, are still very popular in numerical astronomy. Störmer methods of order 13 and 14 aimed at high precision calculations appear in recent research publications like [11, 4], and are implemented in several program packages like the popular NBI-package from the UCLA astronomy group[15]. Granted, these methods are generally not energy or symplecticity preserving, and they thus fall outside the more recent field of geometric numerical integration wherein we find methods that are thought to be more appropriate for mechanical problems. There are however still good reasons to consider non-geometric integrators. In particular if the aim of the computation is very high precision where the right qualitative behaviour comes

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implicitly with the high precision. In recent works involving high precision computations have traditional methods like e.g. Störmer's methods been central (see for example investigations on the stability of the outer solar system [7] and [1]). High order methods with tiny step sizes are used for such computations. However, stability and round-off problems for small step sizes must be carefully analysed and treated in in order for these high precision computations to be truly high precision[15]. The methods are known as prone to rounding errors problems when running with small step sizes. This is described, among other places, in Wanner, Hairer & Nørsett's book[6], where a stabilisation of the scheme is also proposed.

In this work we shall address the question of stability of Störmer-Cowell methods for small step sizes. A basic result here is Dahlquist's classical result from 1976 which bars unconditionally stable methods of order exceeding two[2]. The methods are easily proved to be zero-stable. However, when considering the *absolute stability*, as defined by Lambert[9] we shall see that they are not necessarily stable at any point close to zero. This is strongly related to a well known property of Störmer-Cowell methods, namely that they suffer from what is called orbital instabilities when the order of the method exceeds two; whenever integrating a circular orbit the numerical solution will either spiral inwards or outwards. This has been the background for the development of symmetric multistep methods[10, 13] and methods that integrate trigonometric polynomials exactly[3]. Here we will go back to a more thorough analysis of the stability of the Störmer-Cowell methods and characterize exactly which methods are absolutely stable and unstable for small step sizes. We shall also see examples methods that are unstable for small step sizes, but exhibit stable regions away from zero. Stabilisation schemes will not be discussed in this work.

Let us end this introduction with a few words about the history and timing of the this work. This paper is namely based on a master thesis by Even Thorbergsen from 1976[14] and an unfinished note by Nørsett from the year after. The note was however put in a drawer and forgotten. Recently, while cleaning his office in preparation for his retirement Nørsett found the note, and in discussions with his former PhD student Asheim it was concluded that the note could, in a reworked and finished form, be of interest to the numerical analysis and celestial mechanics community.

2 Multistep methods for second order problems

We start by reviewing some relevant facts regarding multistep methods for second order problems. Full expositions of the following theory are given in books by Henrici[8], Lambert[9] and Wanner, Hairer & Nørsett[6].

We are considering second order initial value problems of the form (1). Multistep methods for such problems are of the general form,

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j}. \tag{2}$$

Defining the linear difference operator

$$\mathcal{L}[y(x);h] = \sum_{j=0}^{k} \left[\alpha_j y(x-jh) - h^2 \beta_j y''(x+jh)\right],$$

we say, using the notation of Henrici[8], that the method has order p if its expansion in h around zero is of the form

$$\mathcal{L}[y(x);h] = C_{p+2}h^{p+2}y^{(p+2)}(x) + C_{p+3}h^{p+3}y^{(p+3)}(x) + \dots, \qquad (3)$$

for any sufficiently smooth $y(x)$. We say that the method is *consistent* if it has order at least one. Defining the generating polynomials,

$$\rho(\zeta) = \sum_{j=0}^{k} \alpha_j \zeta^j, \qquad \sigma(\zeta) = \sum_{j=0}^{k} \beta_m \zeta^j, \qquad (4)$$

this is equivalent with,

$$\rho(1) = \rho'(1) = 0, \qquad \rho''(1) = 2\sigma(1). \qquad (5)$$

Central to this work is two concepts of stability, both which are stated by Lambert[9].

Definition 1. We say that the method (2) is *zero stable* if all roots of $\rho(\zeta)$ are contained in the unit disk, and those roots that are located on the unit circle are of multiplicity at most two.

Definition 2. The method (2) is called *absolutely stable* for a given $q \in \mathbb{C}$ if and only if all the roots of the stability polynomial

$$\varphi(\zeta) := \rho(\zeta) + q^2\sigma(\zeta),$$

are contained in the unit disc. Otherwise we call the method *absolutely unstable* for this q.

2.1 Störmer-Cowell methods

We shall consider multistep methods of the form

$$y_{n+k} - 2y_{n+k-1} + y_{n+k-2} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j}. \qquad (6)$$

These methods are usually referred to as Störmer-Cowell methods. One way to arrive at such methods arise when adding the Taylor series for $y(x_n + h)$ and $y(x_n - h)$,

$$y_{n+1} - 2y_n + y_{n-1} = h^2 y''(x, y_n) + \frac{h^4}{12} y^{(4)}(x, y_n) + \frac{h^6}{360} y^{(6)}(x, y_n) + \dots$$

Störmer's methods are obtained by in this case replacing derivatives of $y(x, y_n)$ with backward differences. Truncating and eliminating higher order terms generates methods of arbitrary order. We define the generating polynomials for these methods,

$$\rho(\zeta) = \zeta^k - 2\zeta^{k-1} + \zeta^{k-2}, \qquad \sigma(\zeta) = \sum_{j=0}^{k} \beta_j \zeta^j. \qquad (7)$$

Adapting the notation from [8] we denote by

$$y_{n+1} - 2y_n + y_{n-1} = h^2 \sum_{j=0}^{k} \sigma_j \nabla^j f_n, \qquad (8)$$

j	0	1	2	3	4	5	6	7
σ_j	1	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{19}{240}$	$\frac{3}{40}$	$\frac{863}{12096}$	$\frac{275}{4032}$
σ_j*	1	-1	$\frac{1}{12}$	0	$-\frac{1}{240}$	$-\frac{1}{240}$	$-\frac{221}{60480}$	$-\frac{19}{6048}$

j	8	9	10	11	12
σ_j	$\frac{33953}{518400}$	$\frac{8183}{129600}$	$\frac{3250433}{53222400}$	$\frac{4671}{78848}$	$\frac{13695779093}{237758976000}$
σ_j*	$-\frac{9829}{3628800}$	$-\frac{407}{172800}$	$-\frac{330157}{159667200}$	$-\frac{24377}{13305600}$	$-\frac{4281164477}{2615348736000}$

Table 1: Coefficients for Störmer's (σ) and Cowell's (σ^*) methods.

the k-step Störmer method of order $k+1$. ∇ is the backwards difference operator: $\nabla z_n = z_n - z_{n-1}$. Likewise we have the k-step *implicit* Cowell-method of order $k+1$,

$$y_{n+1} - 2y_n + y_{n-1} = h^2 \sum_{j=0}^{k} \sigma_j^* \nabla^j f_{n+1}. \tag{9}$$

Translating into ordinate form (6) is achieved by applying the formula

$$\nabla^m f_j = \sum_{l=0}^{m} (-1)^l \binom{m}{l} f_{j-l}.$$

Coefficients of these methods are obtained in a more straightforward way than the above described expansion by interpolation on the right hand side $f(x,y)$, for Störmer's methods yielding

$$\sigma_m = (-1)^m \int_0^1 (1-s) \left[\binom{-s}{m} + \binom{s}{m} \right] \mathrm{ds}, \tag{10}$$

and for Cowell's methods,

$$\sigma_m^* = (-1)^m \int_{-1}^0 (-s) \left[\binom{-s}{m} + \binom{s+2}{m} \right] \mathrm{ds}. \tag{11}$$

Numerical values of coefficients are most conveniently computed by recursion formulas[8]. Table 1 gives values up to $k = 12$.

2.1.1 Some properties of Störmer-Cowell methods

For the following stability analysis we will need some results regarding the methods. The following proposition essentially states that the coefficients of Störmer's methods are positive and decreasing, negative and increasing for Cowell's methods.

Proposition 1. *The coefficients of Störmer's method* (10) *satisfy*

$$\sigma_m \geq 0, \qquad m \geq 1,$$

with equality only for $m = 1$, and

$$\sigma_{m+1} \leq \sigma_m, \qquad m \geq 2,$$

with equality only for $m = 2$. The coefficients of Cowell's method (11) *satisfy*

$$\sigma_{m+1}^* \leq 0, \qquad m \geq 3,$$

with equality only for $m = 3$, and

$$\sigma^*_{m+1} \geq \sigma^*_m, \qquad m \geq 4,$$

with equality only for $m = 4$.

Proof. We shall only do the proof for the case of Störmer's methods. The proof for Cowell's method is completely analogous. Since this proof is based on manipulation of binomial coefficients we will refer to the following identities[12],

$$\binom{t}{m} = \binom{t}{m-1}\frac{t-m+1}{m}, \tag{12}$$

$$\binom{t}{m} = (-1)^m \binom{m-t-1}{m}. \tag{13}$$

The positivity of the coefficients is showed using identity (13) on equation (10),

$$\sigma_m = \int_0^1 (1-s)\left[\binom{m-s-1}{m} + \binom{m+s-1}{m}\right] \mathrm{d}s.$$

Furthermore, using the identity (12) gives

$$\sigma_m = \frac{1}{m}\int_0^1 (1-s)s\left[\binom{m+s-1}{m-1} - \binom{m-s-1}{m-1}\right] \mathrm{d}s. \tag{14}$$

Now defining the function $F(s) = \binom{m+s-1}{m-1}$ we have,

$$\binom{m+s-1}{m-1} - \binom{m-s-1}{m-1} = F(s) - F(-s) = \int_{-s}^{s} \Psi'(s)\mathrm{d}s.$$

Performing the differentiation we get,

$$F'(s) = F(s)\left(\psi(m+s) - \psi(1+s)\right),$$

where $\psi(t)$ is the digamma function. Using that $\psi(t)$ is monotonously increasing for $t > 0$ and $F(s) > 0$ for $s > -1$ and $m > 1$ we see that $\binom{m+s-1}{m-1} - \binom{m-s-1}{m-1}$ is positive, which implies that $\sigma_m > 0$ as $m > 1$. By inspection it is verified that $\sigma_1 = 0$, proving the first part of the proposition.

Using equation (14) we get for the difference of two consecutive coefficients

$$\sigma_m - \sigma_{m+1} = \int_0^1 (1-s)s\left(F(s) - F(-s)\right)\mathrm{d}s,$$

with

$$F(s) = \frac{1}{m+1}\binom{m+s}{m} - \frac{1}{m}\binom{m+s-1}{m-1} = \frac{s-1}{m(m+1)}\binom{m+s-1}{m-1}.$$

where the last equality follows from the recursion $\binom{t}{m} = \binom{t-1}{m-1} + \binom{t-1}{m}$ and identity (12). Applying the identity once more yields,

$$F(s) = \frac{s^2-1}{(m+1)m(m-1)}\hat{F}(s),$$

with

$$\hat{F}(s) = \binom{m+s-1}{m-2},$$

such that

$$\sigma_m - \sigma_{m+1} = \int_0^1 \frac{(1-s)s(s^2-1)}{(m+1)m(m-1)} \left(\hat{F}(s) - \hat{F}(-s)\right) \mathrm{d}s. \qquad (15)$$

Again we have $\hat{F}(s) - \hat{F}(-s) = \int_{-s}^{s} \hat{F}'(s)\mathrm{d}s$. Performing the differentiation yields

$$\hat{F}'(s) = \hat{F}(s)\left(\psi(m+s) - \psi(s+2)\right).$$

From this we see that $\hat{F}(s) - \hat{F}(-s) > 0$ for $m > 2$. Together with equation (15),

$$\sigma_m > \sigma_{m+1}, \quad \text{for} \quad m > 2.$$

By inspection it is verified that $\sigma_3 = \sigma_2$, proving the second part of the proposition.

□

Secondly we need some properties of Störmer's and Cowell's methods related to the order constants (3).

Proposition 2. *The error of a Störmer method of order $p \neq 2$ has an expansion of the form* (3) *where*

$$C_{p+2} = \sigma_p$$
$$C_{p+3} = \frac{p-2}{2}\sigma_p + \sigma_{p+1}.$$

Likewise, for the Cowell method of order $p \neq 4$,

$$C_{p+2} = \sigma_p^*$$
$$C_{p+3} = \frac{p}{2}\sigma_p^* + \sigma_{p+1}^*.$$

Proof. The form of the error and the expression for C_{p+2} is shown in [8]. Let us repeat the argument here: Applying the order $k+1$ Störmer method (8) to $y(x) = x^{k+3}$, the error of the method must necessarily be, since the order $k+2$ method is exact for polynomials of degree $k+3$,

$$\begin{aligned} y_{n+1} - 2y_n + y_{n-1} - h^2 \sum_{j=0}^{k} \sigma_j \nabla^j y'' &= h^2 \sigma_{k+1} \nabla^{k+1} y'' \\ &= h^{k+3}(k+3)!\sigma_{k+1}. \end{aligned}$$

For the last equality we have used that $\nabla^m x^m = h^m m!$. Now comparing with (3) yields the result. In order to find C_{p+3} we will need the identity

$$\nabla^m x^{m+1} = h^m (m+1)! \left(x - \frac{m}{2}h\right), \qquad (16)$$

which is demonstrated by e.g. an induction argument on Leibnitz rule for finite differences. Now the procedure is the same as above. Applying the method (8) to $y(x) = x^{k+4}$, the error is, since the order $k+3$ method is exact for polynomials of degree $k+4$,

$$
\begin{aligned}
y_{n+1} - 2y_n + y_{n-1} - h^2 \sum_{j=0}^{k} \sigma_j \nabla^j y'' =& h^2 \sigma_{k+1} \nabla^{k+1} y'' + h^2 \sigma_{k+2} \nabla^{k+2} y'' \\
=& h^{k+3} (k+4)! \sigma_{k+1} \left(x + \frac{k-1}{2} h \right) \\
&+ h^{k+4} (k+4)! \sigma_{k+2}.
\end{aligned}
$$

Collecting the coefficient of h^{k+3} and comparing with (3) gives

$$
C_{p+3} = \sigma_{k+2} + \sigma_{k+1} \frac{k-1}{2} = \frac{p-2}{2} \sigma_{p+1} + \sigma_p.
$$

Repeating these steps in a straightforward manner yields the stated result for Cowell's methods. □

3 Absolute stability as $h \sim 0$ of Störmer's and Cowell's methods

In this section we shall discuss the stability of the Störmer and Cowell methods in detail. Regarding zero stability, as defined in Definition 1, it is easily seen that all Störmer-Cowell methods satisfy this criterion with a double root at 1 and all other roots at the origin. Note that the double root at 1 is necessary for consistency. We shall see in the following that under small perturbations of q around 0 the double root of $\rho(z)$ will possibly split up into two roots, and the absolute stability of the methods now depend on whether these roots remain within the unit disk or not.

Before discussing the absolute stability of the methods, we shall need some more background on the notion of absolute stability defined in Definition 2. Considering the test equation

$$
y'' = -\lambda y, \tag{17}
$$

with solutions of the form $y(x) = C_1 e^{i\lambda x} + C_2 e^{-i\lambda x}$, we see that solutions are bounded for real λ. Applying the method (2) to this equation, using an ansatz of the form $y(x_0 + nh) \approx y_n = \zeta^n$ leads to the characteristic equation

$$
\varphi(\zeta) = \rho(\zeta) - q^2 \sigma(\zeta) = 0, \qquad q = (\lambda h)^2. \tag{18}
$$

Clearly, a root of magnitude larger than one will lead to a possibly unbounded numerical solution. This lies behind the definition of absolute stability in Definition 2. We will in the following analysis refer both to the test equation and characteristic equation to arrive at the result.

In order to investigate the absolute stability of our methods near zero we clearly have to focus our attention on the double root of the characteristic equation at $q = 0$ and determine how it moves with growing q. Therefore we write

$$
\varphi(r(q)) = 0, \quad r(0) = 1,
$$

and investigate the absolute value of $r(q)$ for small values of q. Now write

$$
r(q) = e^{iq} + g(q),
$$

where we note that $\lim_{q\to 0} g(q) = 0$. Inserting into the characteristic equation (18) and expanding gives

$$\rho(e^{iq})+g(q)\rho'(q^{iq})+\mathcal{O}(g^2)+q^2\sigma(e^{iq})+g(q)q^2\sigma'(e^{iq})+\mathcal{O}(q^2g^2)=0. \quad (19)$$

In the following we have to work with this equation for odd and even orders separately.

3.1 Odd orders

Assuming the method is of order $p = 2m+1$ with $m \geq 1$, the there holds, using (3) with solutions of (17)

$$\begin{aligned}\rho(e^{iq})+q^2\sigma(e^{iq}) &= C_{p+2}h^{p+2}(i\lambda)^{p+2}+C_{p+3}h^{p+3}(i\lambda)^{p+3}+\mathcal{O}(q^{p+4})\\ &= i(-1)^{m+1}C_{p+2}q^{p+2}+(-1)^mC_{p+3}q^{p+3}+\mathcal{O}(q^{p+4}).\end{aligned}$$

Inserting into (19) gives

$$\begin{aligned}&\left[\rho'(e^{iq})+q^2\sigma'(e^{iq})\right]g(q)+\mathcal{O}(g^2)(1+q^2)=\\ &\qquad (-1)^{m+1}C_{p+3}q^{p+3}+i(-1)^mC_{p+2}q^{p+2}+\mathcal{O}(q^{p+4}).\end{aligned} \quad (20)$$

Comparing orders of q gives that $g(q) = \mathcal{O}(q^{p+2})$. Using the consistency of the method we get that,

$$\rho'(e^{iq}) = \rho''(1)qi - \frac{1}{2}(\rho''(1)+\rho'''(1))q^2+\mathcal{O}(q^3). \quad (21)$$

Likewise,

$$\sigma'(e^{iq}) = \sigma''(1)qi + \left[\sigma'(1)-\frac{1}{2}(\sigma''(1)+\sigma'''(1))q^2\right]+\mathcal{O}(q^3). \quad (22)$$

Therefore we get

$$\rho'(e^{iq})+q^2\sigma'(e^{iq}) = a_0q^2+a_1qi+\mathcal{O}(q^3), \quad (23)$$

where

$$a_0 = -\frac{1}{2}(\rho''(1)+\rho'''(1))+\sigma'(1), \qquad a_1 = \rho''(1). \quad (24)$$

Now isolate the real and imaginary parts of $g(q)$, $g(q) = g_0(q)+ig_1(q)$, substitute into (20), and isolate real and imaginary parts of the equation(noting that C_{p+2} and C_{p+3} are real by Proposition 1),

$$\begin{aligned}a_0g_0(q)q^2 - a_1g_1(q)q &= (-1)^{m+1}C_{p+3}q^{p+3}+\mathcal{O}(q^{p+4}), \quad (25)\\ a_1g_0(q)q+a_0q_1(q)q^2 &= (-1)^mC_{p+2}q^{p+2}+\mathcal{O}(q^{p+3}).\end{aligned}$$

Again comparing orders of q shows that $g_0(q) = \mathcal{O}(q^{p+1})$ and $g_1(q) = \mathcal{O}(q^{p+2})$. Eliminating higher order terms and solving for $g_0(q)$ gives,

$$g_0(q) = (-1)^m\frac{C_{p+2}}{\rho''(1)}q^{p+1}+\mathcal{O}(q^{p+2}). \quad (26)$$

Now we are in position to investigate the size of $r(q)$ under small perturbations,

$$|r(q)| = 1+|g(q)|^2+2\Re[e^{-iq}g(q)] = 1+2g_0(q)+\mathcal{O}(q^{p+2}).$$

The final step is valid since,

$$\Re[e^{-iq}g(q)] = \Re[(1 - iq + \ldots)g(q)] = g_0(q) - g_1(q)q + \mathcal{O}(gq^2),$$

and $g_1(q) = \mathcal{O}(q^{p+2})$. The absolute stability of the method for small q can thus be characterized in terms of the sign of the function $A_o(p,q)$,

$$A_o(m,\rho) = 2\frac{(-1)^m C_{2m+3}}{\rho''(1)}. \tag{27}$$

This will be made explicit in Theorem 1. This theorem also includes the case of even orders, which will be investigated in the following.

3.2 Even orders

We proceed analogously for even orders. Assuming $p = 2m$, $m \geq 2$, then the there holds,

$$\begin{aligned}\rho(e^{iq}) + q^2\sigma(e^{iq}) &= C_{p+2}h^{p+2}(i\lambda)^{p+2} + C_{p+3}h^{p+3}(i\lambda)^{p+3} + \mathcal{O}(q^{p+4})\\ &= (-1)^{m+1}C_{p+2}q^{p+2} + i(-1)^{m+1}C_{p+3}q^{p+3} + \mathcal{O}(q^{p+4}).\end{aligned}$$

Following the steps (21), (22), (23), with $g(q) = g_0(q) + ig_1(q)$, leads to the system

$$\begin{aligned}a_0g_0(q)q^2 - a_1g_1(q)q &= (-1)^m C_{p+2}q^{p+2} + \mathcal{O}(q^{p+3}),\\ a_1g_0(q)q + a_0g_1(q)q^2 &= (-1)^m C_{p+3}q^{p+3} + \mathcal{O}(q^{p+4}).\end{aligned} \tag{28}$$

Comparing orders of q shows now that $g_0(q) = \mathcal{O}(q^{p+2})$ and $g_1(q) = \mathcal{O}(q^{p+1})$. Eliminating higher order terms and solving for $g_0(q)$ and $g_1(q)$ gives,

$$\begin{aligned}g_1(q) =& (-1)^{m+1}\frac{C_{p+2}}{\rho''(1)}q^{p+1} + \mathcal{O}(q^{p+2}), \qquad (29)\\ g_0(q) =& \frac{(-1)^m}{\rho''(1)^2}\left[\rho''(1)C_{p+3} - \frac{1}{2}C_{p+2}\left(\rho''(1) + \rho'''(1) - 2\sigma'(1)\right)\right]q^{p+2}\\ &+ \mathcal{O}(q^{p+3}).\end{aligned}$$

Order conditions for order 2 [8] requires that

$$\rho''(1) = 2\sigma(1), \quad \text{and} \quad \rho'''(1) = 6\sigma'(1) - 6\sigma(1). \tag{30}$$

Thus, inserting (30) into (29), we get

$$g_0(q) = \frac{(-1)^m}{\rho''(1)^2}\left[\rho''(1)C_{p+3} - \frac{1}{3}\rho'''(1)C_{p+2}\right]q^{p+2} + \mathcal{O}(q^{p+3}).$$

Again investigating the size of the roots leads to

$$|r(q)| = 1 + |g(q)| + 2\Re[e^{-iq}g(q)] = 1 + A_e(m,\rho)q^{p+2} + \mathcal{O}(q^{p+3}),$$

where

$$A_e(m,\rho) = 2\frac{(-1)^m}{\rho''(1)^2}\left[\rho''(1)C_{2m+3} - \left(\frac{1}{3}\rho'''(1) + \rho''(1)\right)C_{2m+2}\right]. \tag{31}$$

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Störmer	✓	✓	✓	✓	0	0	✓	✓	0	0	✓	✓	0	0
Cowell	✓	✓	✓	✓	✓	✓	0	0	✓	✓	0	0	✓	✓

Table 2: Methods that are stable in the vicinity of zero: ✓denotes stable method, 0 - unstable.

3.3 The result

We now synthesize the main result of this paper, the stability Störmer and Cowell methods.

Theorem 1.

1. *The k-step Störmer method of order $p = k+1$ (8) is absolutely stable for $q \sim 0$, $q \neq 0$, whenever $p = 4l - 1$ or $p = 4l$, and absolutely unstable whenever $p = 4l + 1$ or $p = 4l + 2$, $l = 1, 2, \ldots$.*
2. *The k-step Cowell method of order $p = k+1$ (9) is absolutely unstable for $q \sim 0$, $q \neq 0$, whenever $p = 4l-1$ or $p = 4l$, and absolutely stable whenever $p = 4l + 1$ or $p = 4l + 2$, $l = 1, 2, \ldots$. $l = 1, 2, \ldots$.*

Proof. In this proof we will need the following regarding the characteristic polynomial $\rho(\zeta)$, defined in equation (7)

$$\rho''(1) = 2, \quad \text{and} \quad \frac{1}{3}\rho'''(1) + \rho''(1) = 2(k-1). \tag{32}$$

Both equalities are easily verified by straightforward calculations.

1. For Störmer's method we have from Propositions 1 and 2 that $C_{p+2} = \sigma_{p+1} > 0$. Therefore, investigating the sign of $A_o(m, \rho)$ defined in equation (27), using equation (32), we see that the method will be absolutely stable for $q \sim 0$, $q > 0$, whenever m is odd; $m = 2l-1$, $l = 1, 2, \ldots$. This corresponds to order $p = 2m+1 = 4l-1$. Likewise the method is unstable for even m; $m = 2l$, $l = 1, 2 \ldots$. This corresponds to order $p = 2m + 1 = 4l + 1$.

   For even orders we investigate the function $A_e(m, \rho)$ defined in equation (31). Using proposition 2 and equation (32), we have for Störmer's methods,

   $$\begin{aligned} A_e(m, \rho) &= (-1)^m \left(C_{p+3} - (p-2)C_{p+2}\right) \\ &= (-1)^m \left(\sigma_{p+2} - \frac{p-2}{2}\sigma_{p+1}\right). \end{aligned} \tag{33}$$

   Now Proposition 1 guarantees that the factor $\sigma_{p+2} - \frac{(p-2)}{2}\sigma_{p+1}$ is negative as long as $p > 3$. Using this, we see that the method is absolutely stable when $q \sim 0$, $q > 0$, if m is even, $m = 2l$, $l = 1, 2, \ldots$, corresponding to order $p = 2m = 4l$. Likewise will the method be absolutely unstable, $q \sim 0$, $q > 0$, if $p = 4l + 2$.

2. For Cowell's method we repeat the exact same argument as for Störmer's, but with reversed signs. This gives that, provided $p > 3$, the method is absolutely unstable in the vicinity of zero whenever Störmer's method is absolutely stable and vice versa.

□

Thus we have established the stability of the Störmer's methods and Cowell's methods for $k > 3$. For smaller k stability is checked case by case. This will be done more in detail in the following section where we shall establish intervals of stability for some of the lower order methods. We sum up the stability of the methods in the vicinity of zero in Table 2.

4 Regions of absolute stability

In order to visualize the actual regions of absolute stability we use what is known as *root-locus curve* in the classical theory of multistep methods[6]. The root-locus curve in the case of the method (6) is simply the image of the unit circle under the transformation $z \to -\sigma(z)/\varphi(z)$. The importance of this curve lies in the fact that the boundary of the region of absolute stability will necessarily be a subset of this curve.

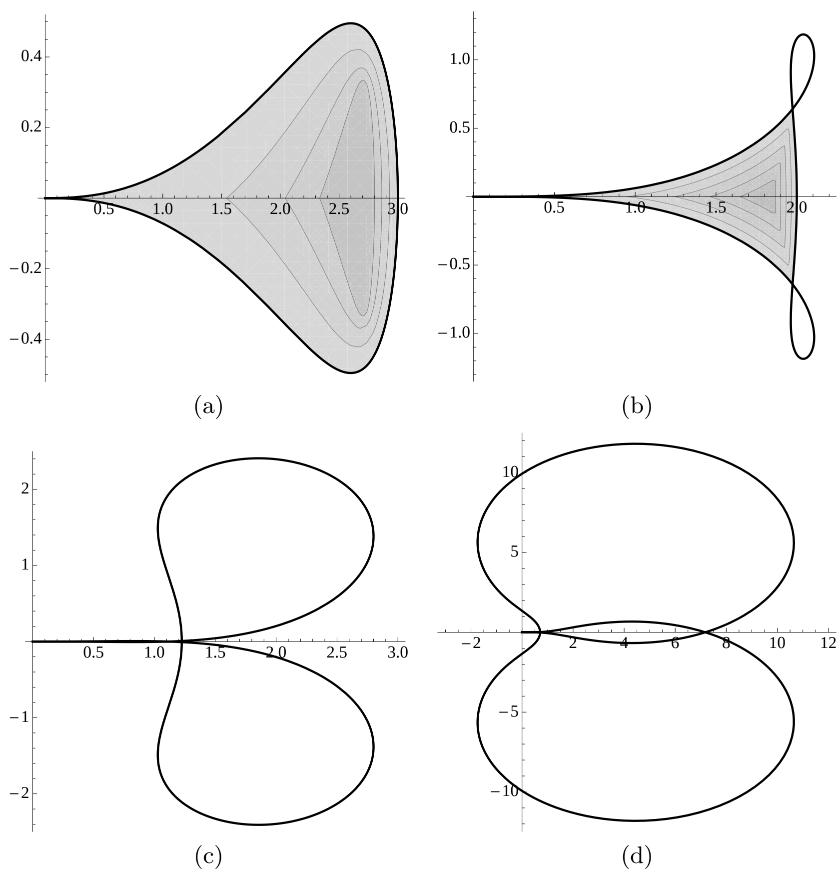

Figure 1: The Root-Locus curve and stability regions for Störmer's methods: (a) $k = 2$, (b) $k = 3$, (c) $k = 4$, (d) $k = 5$.

Starting with Störmer's method, we know that $k = 0$ and $k = 1$ both correspond to the order 2 Störmer-Verlet method. It is easily verified that this method is absolutely stable for $|q| < 2$. For $k = 2, 3, 4, 5$ we draw the root-locus curves, and determine the stability regions case by case. Figure

k	Störmer	Cowell
0	$[-2, 2]$	$[0, \infty]$
1	$[-2, 2]$	$[0, 4]$
2	$[0, 3]$	$[0, 6]$
3	$[0, 2]$	$[0, 6]$
4	$[1.114\ldots, \frac{60}{49}]$	$[0, \frac{60}{11}]$
5	unstable	$[0, \frac{60}{13}]$
6	$[0, \frac{378}{967}]$	$[0.9314, \frac{189}{52}]$
7	$[0, \frac{27}{128}]$	$[2.3136\ldots, \frac{189}{71}]$
8	unstable	$[0, 0.3597\ldots]$
9	unstable	$[0, 1.0218\ldots]$
10	$[0, \frac{51975}{1696934}]$	$[0.1899\ldots, \frac{20790}{28687}]$
11	$[0, \frac{9450}{595163}]$	unstable
12	unstable	$[0, 0.1170\ldots]$
13	unstable	$[0, \frac{232186500}{1628120447}]$

Table 3: Stability intervals for Störmer's and Cowell's methods.

1 shows the result of these calculations. Note that in the case $k = 4$ and $k = 5$, there is no apparent region of stability. However, in the case of $k = 4$ we can zoom in and verify that there is in fact a small region of stability around $q \approx 1.2$ something that might come as a slight surprise, see Figure 3. For $k = 5$ the same kind of investigation shows that there is indeed no regions of stability.

For Cowell's methods it can be verified in a similar case by case investigations that the methods $k = 0, 1, 2, 3$ are stable near zero. In Figure 2 we plot root-locus curves for the methods $k = 4, 5, 6, 7$. In the case of Cowell $k = 7$, there appears to be no region of stability. Again, as in the case of Störmer $k = 4$, by zooming in it is verified that there is in fact a small region of stability away from zero. For Cowell's method with $k = 6$ one can be misled by Figure 2c) to believe that the method is stable near zero. However, by zooming, as shown in Figure 3b) we see that it has a region of stability with lower real limit close to one.

Thus, by carefully examining case by case we can obtain real intervals of stability for higher order methods. This is done in a numerically satisfactory way by finding all points where the root locus curve crosses the real axis, and then test the size of all roots in between to determine if the corresponding intervals are stable or unstable. The result for k up to 13 is listed in Table 3.

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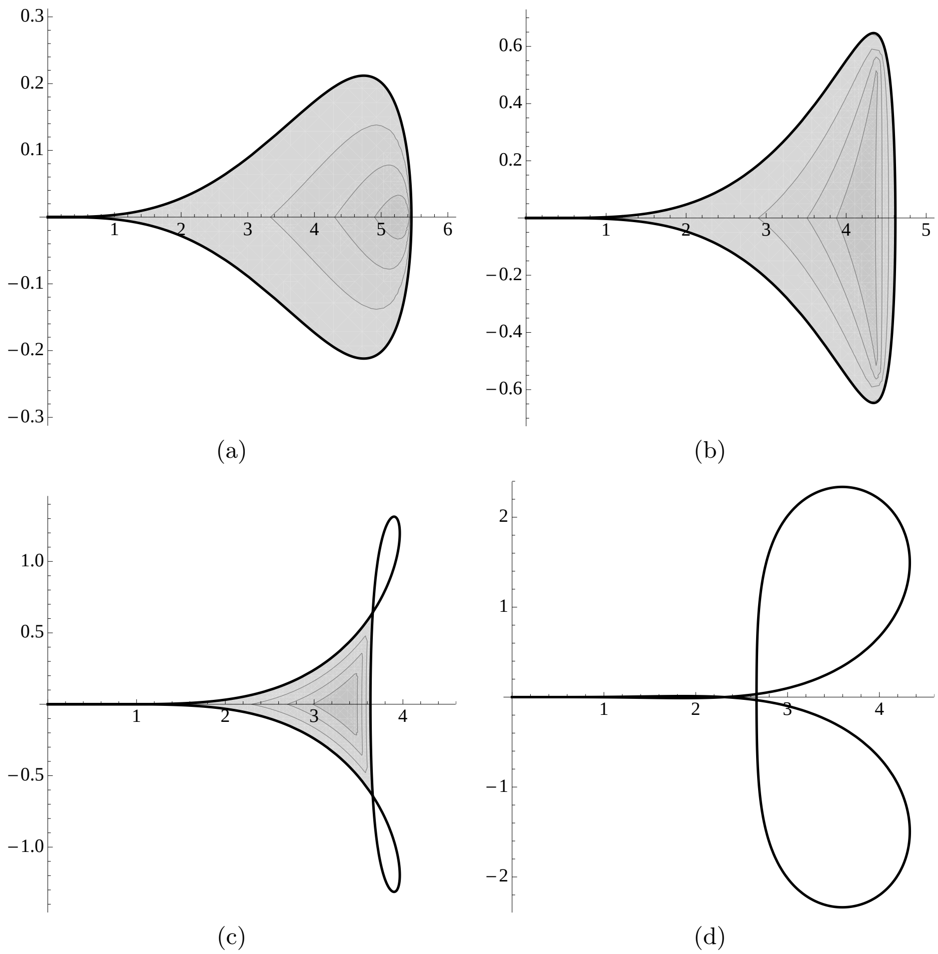

Figure 2: The Root-Locus curve and stability regions for Cowell's methods: (a) $k = 4$, (b) $k = 5$, (c) $k = 6$, (d) $k = 7$.

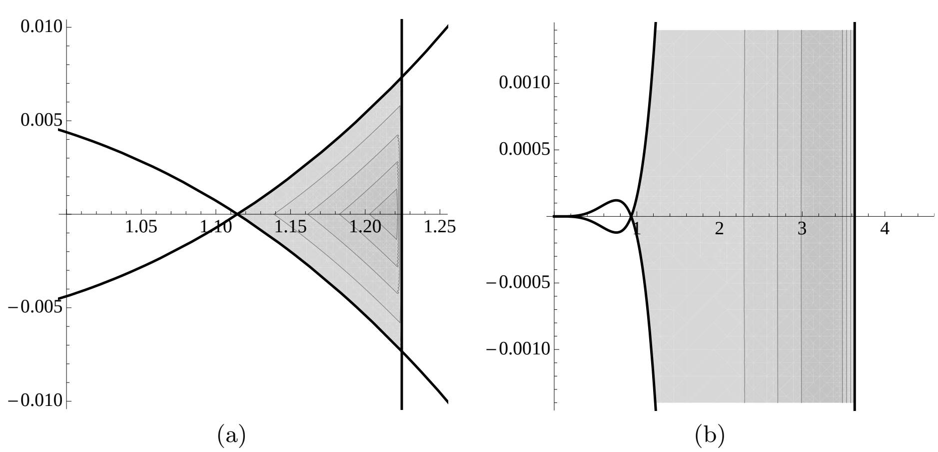

Figure 3: Zoom-in on stability regions for (a) Störmer's method with $k = 4$, (b) Cowell's method with $k = 7$.

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